Counting Integral Lamé Equations by Means of Dessins d'Enfants

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Abstract

We obtain an explicit formula for the number of Lamé equations (modulo scalar equivalence) with index n and projective monodromy group of order $2N$, for given $n \in \mathbb{Z}$ and $N \in \mathbb{N}$. This is done by performing the combinatorics of the 'dessins d'enfants' associated to the Belyi covers which transform hypergeometric equations into Lamé equations by pull-back.

1 Introduction

The integral Lamé equation with parameters $B, g_2, g_3 \in \mathbb{C}$ and $n \in \mathbb{Z}$ is the second order differential equation on the $\mathbb{P}^1$ given by

$$p(z)\frac{d^2y}{dz^2} + \frac{1}{2}p'(z)\frac{dy}{dz} - (n(n+1)z + B)y = 0, \tag{1}$$

where $p(z) := 4z^3 - g_2 z - g_3$ has nonzero discriminant, i.e. $g_2^3 - 27g_3^2 \neq 0$. The parameter n is called the *index*. We are interested in Lamé equations with a basis of solutions which are algebraic (over $\mathbb{C}(z)$). Such solutions can in fact occur for non integer $n \in \mathbb{Q}$, but in this article we restrict to $n \in \mathbb{Z}$, hence the word integral in 'integral Lamé equation'.

It is known that if there exists a basis of algebraic solutions, then the projective monodromy group is dihedral, see for example [BW, Cor. 3.4] or [vdW, Thm. 4.5.5]. If in equation (1) we replace z by λz (for a nonzero $\lambda \in \mathbb{C}$), we get a new Lamé equation with the same index n, but with the parameters (B, g_2, g_3) replaced by $(B/\lambda, g_2/\lambda^2, g_3/\lambda^3)$. These substitutions induce a natural equivalence relation on the space of all Lamé equations. Two Lamé equations which are equivalent w.r.t. this equivalence relation are called *scalar equivalent*. Now it is also known that for given $n \in \mathbb{Z}$ and $N \in \mathbb{N}$ there are only finitely many Lamé equations modular scalar equivalence with index n and projective monodromy group dihedral of order $2N$. This can be proven in different ways, see for example [BW, Thm. 4.6], [vdW, Thms. 5.4.4, 6.7.9] or [Lit1, Thm 4.1]. Throughout this article the number of Lamé equations modular scalar equivalence with index n and projective monodromy group dihedral of order $2N$ will be denoted by $L(n, N)$.

In [Lit1] it is described how the problem of calculating $L(n, N)$ can be translated in counting the number of dessins d'enfants compatible with prescribed ramification data. In the same article the combinatorics are performed for $n = 1$, obtaining a result earlier obtained in [Chi]. In [Lit2] an attempt was made to perform the combinatorics for $n = 2$. In this article we perform the combinatorics for general $n \in \mathbb{N}$.

2 The Combinatorics

Following [Lit1], we write the Lamé equation as in equation 1, but now with $p(z) := 4z(z-1)(z-\lambda)$ and $\lambda \in \mathbb{C} - \{0, 1\}$. According to [Chi] the functions (Belyi covers) $F : \mathbb{P}^1 \to \mathbb{P}^1$ which transform, by pull-back, hypergeometric equations into Lamé equations with index n and projective monodromy group dihedral of order $2N$, have the following ramification data.

	0	1	λ	∞	
1					$+nN/2$ points with multiplicity 2
∞					$+n$ points with multiplicity N
0	1	1	1	2n+1	$+(nN-2n-4)/2$ points with multiplicity 2

Table 1: Case Ia

	0	1	λ	∞	
1			1		$+(nN-1)/2$ points with multiplicity 2
∞					$+n$ points with multiplicity N
0	1	1		2n+1	$+(nN-2n-3)/2$ points with multiplicity 2

Table 2: Case Ib

	0	1	λ	∞	
1		1	1		$+(nN-2)/2$ points with multiplicity 2
∞					$+n$ points with multiplicity N
0	1			2n+1	$+(nN-2n-2)/2$ points with multiplicity 2

Table 3: Case Ic

	0	1	λ	∞	
1	1	1	1		$+(nN-3)/2$ points with multiplicity 2
∞					$+n$ points with multiplicity N
0				2n+1	$+(nN-2n-1)/2$ points with multiplicity 2

Table 4: Case Id

	0	1	λ	∞	
1					$+nN/2$ points with multiplicity 2
∞		N/2	N/2		$+n-1$ points with multiplicity N
0	1			2n+1	$+(nN-2n-2)/2$ points with multiplicity 2

Table 5: Case II

The tables have the following meaning. All the possible branching points, i.e. $0, 1$ and ∞, are contained in the first column. In the three corresponding rows, the inverse images (which are given in the first row) together with the multiplicities can be read of in the obvious way. For more information we refer to [Lit1] or [Lit2].

Theorem 1 *Let $n, N \in \mathbb{N}$, then the number of dessins d'enfants compatible with the tables above equals $n(n+1)(N-1)(N-2)/12 + 2/3\varepsilon(n, N)$, where*

$$\varepsilon(n, N) := \begin{cases} 1 & \textit{if } 3|N \textit{ and } n \equiv 1 \pmod 3; \\ 0 & \textit{otherwise.} \end{cases}$$

Proof Let $n, N \in \mathbb{N}$ and suppose $n > 1$. First consider case I, it will not be necessary to consider the four cases Ia, Ib, Ic, Id separately. In all four cases there are n points above ∞, all with multiplicity N. So the associated dessins consist of n cells all with valency N. Furthermore, there are $n-1$ cycles, which have beginning and end in the point $p := \infty$ and 3 lines, emanating from p. There are no further intersections. In this article, whenever we draw dessins d'enfants, we will not draw the vertices.

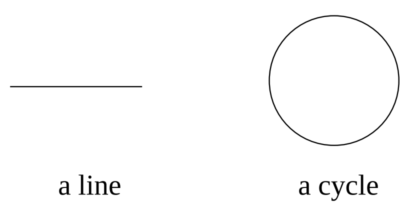

Figure 1: a line and a cycle, not taken into account the valencies

We define the valency of a line as the number of edges of that line and the valency of a cycle as half the number of edges of that cycle. With these definitions we define the valency of a cell as the sum of the valencies of the lines it contains and the cycles that bound it. We note that the valency of a cell equals the multiplicity of the inverse image of ∞ lying in the cell. We see that the boundary of every cell contains exactly one ore two cycles (since we assumed $n > 1$). A cell with exactly one cycle in its boundary will be called a *simple cell*. Since all the cells have the same valency and there are only three lines, there can be no more than three simple cells. There is of course a minimum of two simple cells. These two

cases, that of two simple cells (case A) and that of three simple cells (case B), will be considered seperately.
We first consider case A. If we do not take into account the 3 lines and the valencies of the cycles, it is easy to see that there is only one possibility for the dessin. It has the following shape:

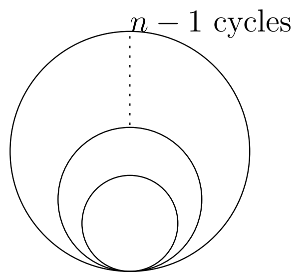

Figure 2: case A

This can also be drawn in a more symmetric way as follows:

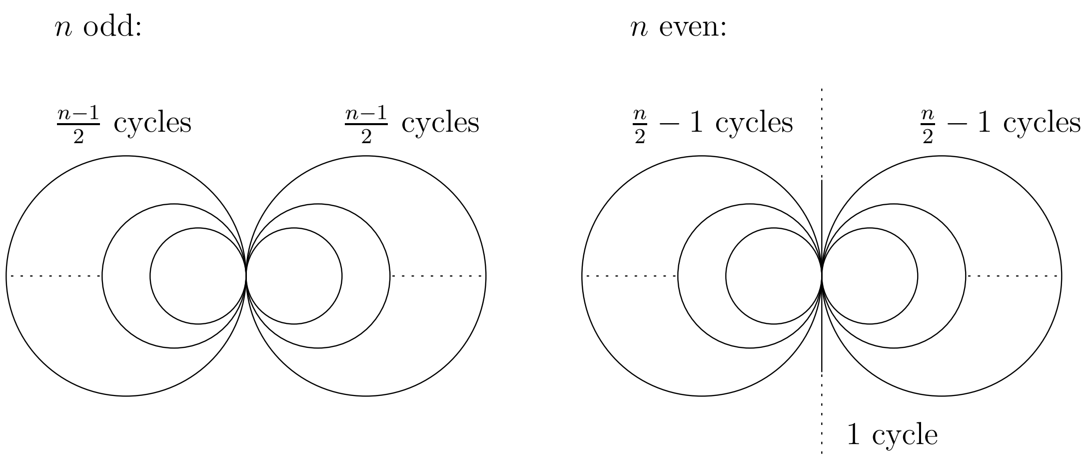

Figure 3: case A

We will now take into account the three lines (but still do not take into account the valencies). Each of the two simple cells must contain at least one line (since every cell has the same valency). If each of the three lines are contained in a simple cell, there is (because of rotational symmetry) only one possibility. We call this case AI. The dessin has the following shape:

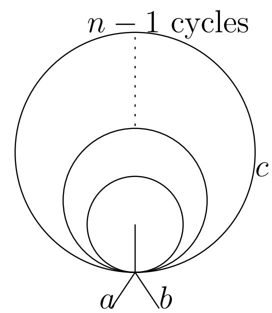

Figure 4: case AI

If there is a line which is not contained in a simple cell, there are $n-2$ possible cells that can contain it. For every of these $n-2$ cells the line can lie on two different sides, but if we take into account the rotational symmetry, we see that the number of possibilities is reduced by a factor two. So in this case we arrive at exactly $n-2$ different kinds of dessins. We call this case AII. The dessins are of the following shape:

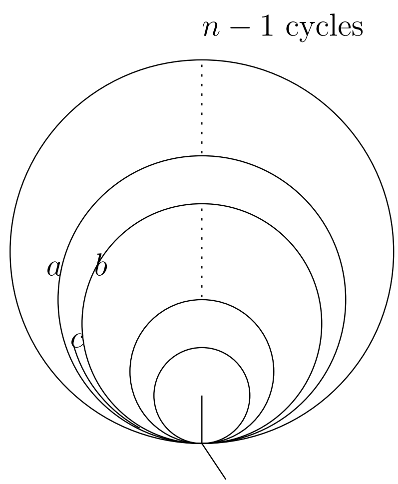

Figure 5: case AII

where the line which is not contained in a simple cell can always be drawn on the left side.
We will now take into account the valencies of the cycles and lines. Let $a, b, c \in \mathbb{N}$ denote the valencies as given in figures 4 and 5. In both cases we have $a+b+c = N$ (and $a, b, c \geq 1$). Furthermore, it is easy to see (by induction) that for every such triple (a, b, c) there is exactly one possible dessin (for a fixed shape of the

dessin). Now there are $(N-1)(N-2)/2$ possible triples (a,b,c). We conclude that there are exactly $(N-1)(N-2)/2$ possible dessins of type AI and there are exactly $(n-2)(N-1)(N-2)/2$ possible dessins of type AII. So there are exactly $(n-1)(N-1)(N-2)/2$ possible dessins of type A.
Now consider case B. If we do not take into account the valencies of the cells and lines, one easily obtains (by induction) that the dessins are of the following shape:

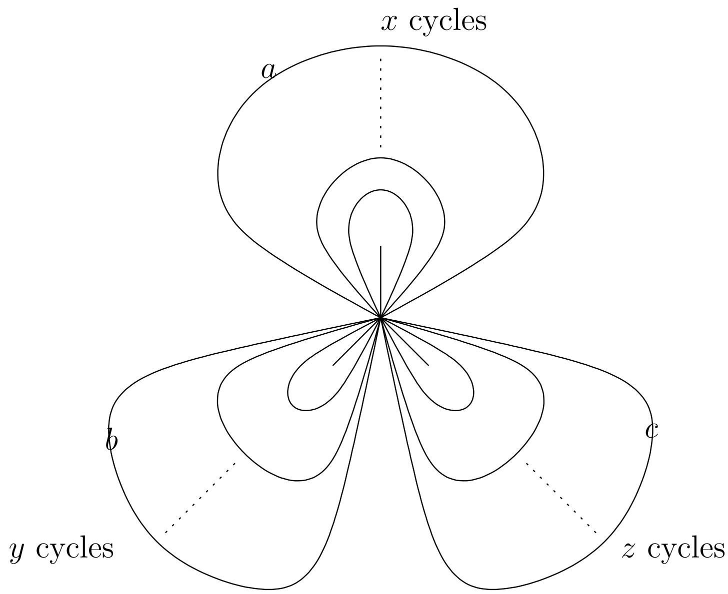

Figure 6: case B

We first consider these dessins without taking into account rotational symmetry. Since $x+y+z=n-1$ (and $x,y,z \geq 1$) there are $(n-2)(n-3)/2$ possibilities for the triple (x,y,z). Note that for $n=2,3$ we also arrive at the correct answer, namely zero. We will now take into account the valencies of the cycles and lines. Let $a,b,c \in \mathbb{N}$ denote the valencies as given in figure 6. We have $a+b+c=N$ (and $a,b,c \geq 1$). Furthermore, it is easy to see (by induction) that for every such triple (a,b,c) there is exactly one possible dessin (for fixed x,y,z and not taken into account the rotational symmetry yet). Now there are $(N-1)(N-2)/2$ possible triples (a,b,c). We conclude that there are $(n-2)(n-3)(N-1)(N-2)/4$ possible dessins of type B, not taken into account the rotational symmetry.
If we now do take into account the rotational symmetry, we see that we have counted every dessin three times, except when the dessin has a three fold rotational symmetry. There is exactly one dessin with a three fold rotational symmetry if and only if N and the number of cycles are divisible by 3, i.e. $3|N$ and $n \equiv 1 \pmod 3$, otherwise there are no such dessins. We conclude that there are

exactly

$$\frac{1}{3}\left(\frac{(n-2)(n-3)(N-1)(N-2)}{4}+2\varepsilon(n,N)\right)=\frac{(n-2)(n-3)(N-1)(N-2)}{12}+\frac{2}{3}\varepsilon(n,N)$$

possible dessins of type B.
So the total number of dessins of type I equals $(n-1)(N-1)(N-2)/2+(n-2)(n-3)(N-1)(N-2)/12+2/3\varepsilon(n,N) = n(n+1)(N-1)(N-2)/12+2/3\varepsilon(n,N)$.
It remains to be proven that there are no dessins of type II. In case II there are $n+1$ points above ∞, $n-1$ with multiplicity N and two with multiplicity $N/2$. So the associated dessins consist of $n+1$ cells, $n-1$ with multiplicity N and two with multiplicity $N/2$. Furthermore, there are n cycles and 1 line, which emanate from $p=\infty$. There are no further intersections. From the fact that there are two cells with valency strictly smaller than N, all other cells have valency N and there is one line, we conclude that (again) there are either 2 simple cells (case C) or 3 simple cells (case D).
First consider the case of two simple cells. We distinguish the following two cases. Case CI: the line is contained in a simple cell. Case CII: the line is not contained in a simple cell.

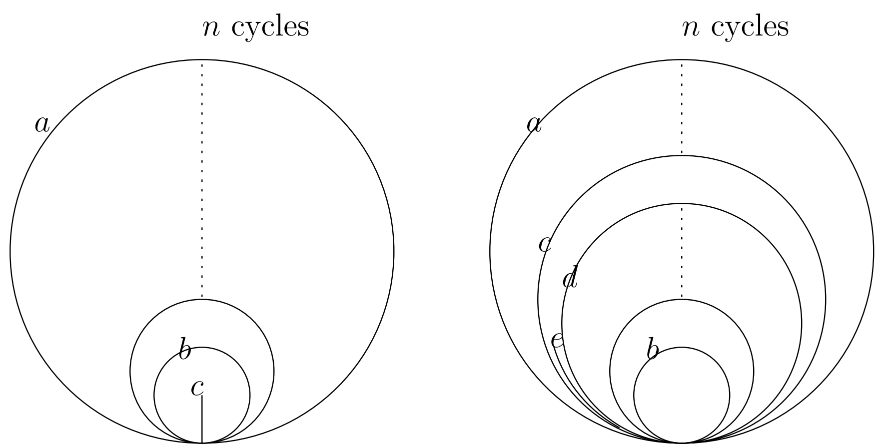

Figure 7: left: case CI, right: case CII

Let $a,b,c \in \mathbb{N}$ denote the valencies as given in the left part of figure 7. We must have $a=N/2$ and by induction we obtain that all the non simple cells have valency N and that $b=N/2$. So $b+c>N/2$, but this means that there is only one cell with valency $N/2$. We conclude that there are no dessins of type CI.
Let $a,b,c,d,e \in \mathbb{N}$ denote the valencies as given in the right part of figure 7. We must have $a=b=N/2$. Since all non simple cells must have valency N, we obtain by induction that $c=d=N/2$. So $c+d+e>N$, but this means that there is a cell with valency larger that N. We conclude that there are no dessins of type CII.

Now consider the case of three simple cells. The line must lie in a simple cell and the dessin has the following shape:

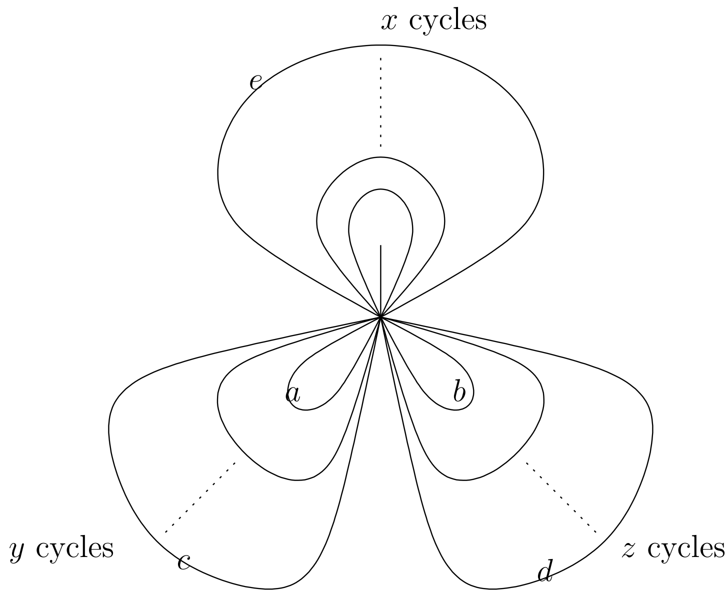

Figure 8: case D

Let $a, b, c, d, e \in \mathbb{N}$ denote the valencies as given in figure 8. We must have $a = b = N/2$. Since all non simple cells must have valency N we obtain by induction that $c = d = N/2$. So $c + d + e > N$, but this means that there is a cell with valency larger that N. We conclude that there are also no dessins of type D. So there are no dessins of type II. We note that for $n = 1$ the proof that there are no dessins of type II also applies (but only case CI has to be considered and there is only one cycle).
We have proven our theorem for $n > 1$. For $n = 1$ the combinatorics was done in [Lit1] (and for $n = 1$ the result was already obtained in [Chi] by other means), but the combinatorics can be simplified significantly as follows.
As noted before there are no dessins corresponding to type II. So let us consider case I. There is only one point above ∞, which has multiplicity N. So the associated dessins consist of one cell with valency N. Furthermore there are no cycles and three lines, who come together in the point $p = \infty$. The dessin has the following shape:

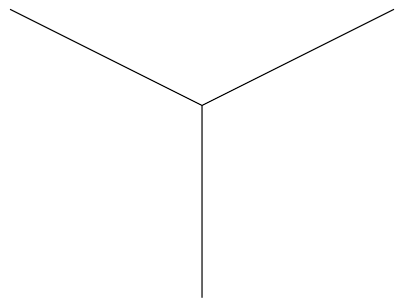

Figure 9: $n = 1$

Let a, b, c denote the valencies of the three lines. If we do not take into account the rotational symmetry, then the number of dessins equals the number of triples (a, b, c) with $a + b + c = N$ (and $a, b, c \geq 1$). There are $(N-1)(N-2)/2$ such triples.
If we now do take into account the rotational symmetry, we see that we have counted every dessin three times, except when the dessin has a three fold rotational symmetry. There is exactly one dessin with a three fold rotational symmetry if and only if $3|N$, otherwise there are no such dessins. We conclude that there are exactly

$$\frac{1}{3}\left(\frac{(N-1)(N-2)}{2} + 2\varepsilon(1, N)\right) = \frac{(N-1)(N-2)}{6} + \frac{2}{3}\varepsilon(1, N)$$

possible dessins when $n = 1$. This finishes the proof. $\square$

Now theorem 1 combined with proposition 3.1 in [Lit1] gives rise to the following.

Theorem 2 *Let $n \in \mathbb{Z}$ and $N \in \mathbb{N}$. Then*

$$\sum_{d|N} L(n, d) = \frac{n(n+1)}{12}(N-1)(N-2) + \frac{2}{3}\varepsilon(n, N), \tag{2}$$

where $\varepsilon(n, N)$ is as in theorem 1.

Let $N \in \mathbb{N}$. We denote by $\phi(N)$ Euler's totient function, i.e.

$$\phi(N) := |\{k \in \mathbb{Z} \mid 0 \leq k < N,\ \gcd(k, N) = 1\}|$$

and by $\Psi(N)$ the two dimensional analog, i.e.

$$\Psi(N) := |\{(k_1, k_2) \in \mathbb{Z}^2 \mid 0 \leq k_1, k_2 < N,\ \gcd(k_1, k_2, N) = 1\}|.$$

From the theorem above, together with the well known results

$$\sum_{d|N} \phi(d) = N \text{ and } \sum_{d|N} \Psi(d) = N^2,$$

we easily obtain the following.

Corollary 3 *Let* $n \in \mathbb{Z}$ *and* $N \in \mathbb{N}$*. If* $N = 1$*, then* $L(n, N) = 0$*. If* $N \neq 1$*, then*

$$L(n, N) = \frac{n(n+1)}{12}(\Psi(N) - 3\phi(N)) + \frac{2}{3}\epsilon(n, N),$$

where

$$\epsilon(n, N) := \begin{cases} 1 & \text{if } N = 3 \text{ and } n \equiv 1 \pmod 3; \\ 0 & \text{otherwise.} \end{cases}$$

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